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On large-amplitude magnetohydrodynamics

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Three results on the flow of an infinitely conducting and inviscid fluid are presented in this paper. The first result is that all steady flows with magnetic lines coincident with streamlines are reducible to flows without a magnetic field. The second result is on the establishment of a steady irrotational and current-free flow with coincident streamlines and magnetic lines. It throws some light on the controversy between Stewartson (1960) and Sears & Resler (1959) concerning the possibility of such a flow. The third result concerns the flow of a fluid through a circular cylinder of radius R into a point sink with strength m when the fluid carries a current of density j_0 at infinity. It is shown that the condition of uniform flow at infinity is impossible to maintain if a dimensionless number $(kR)^2$ involving the current density j_0 exceeds the value $(3{\cdot}831)^2$, and that the current has the effect of concentrating the flow near the centre line and of producing ring eddies which become longer and longer as $(kR)^2$ is increased. The dimensionless number $(kR)^2$ is defined to be

$$|2(\omega-\alpha\gamma)R/W(1-\alpha^2)|^2$$

with $\alpha = h_0/W, \quad \gamma = 2\pi j_0(\mu/4\pi\rho)^{\frac{1}{2}} \quad \text{and} \quad h_0 = H_0(\mu/4\pi\rho)^{\frac{1}{2}},$

in which ω and W are the uniform angular velocity and the velocity at infinity, H_0 is the uniform longitudinal magnetic field at infinity, ρ the density, μ the magnetic permeability, and j_0 the current density at infinity.

1. Preliminaries

This paper deals with the flow of an inviscid, incompressible, and infinitely conducting fluid in a magnetic field. Except in §4, Cartesian co-ordinates x_i $(i = 1, 2, 3)$ will be used. The corresponding velocity components will be denoted by u_i and the corresponding magnetic field components by H_i. As usual, the density of the fluid will be denoted by ρ, the pressure by p, the magnetic permeability by μ, the body force per unit mass by X_i, and the time by t. The magnetic permeability is assumed constant. The equations of motion are

$$\rho\frac{Du_i}{Dt} = -\frac{\partial}{\partial x_i}\left(p+\frac{\mu}{8\pi}H^2\right)+\rho X_i+\frac{\mu}{4\pi}H_\alpha\frac{\partial H_i}{\partial x_\alpha}, \tag{1}$$

and the magnetic-field equations are

$$\frac{DH_i}{Dt} = H_\alpha\frac{\partial u_i}{\partial x_\alpha}, \tag{2}$$

in which $\frac{D}{Dt} = \frac{\partial}{\partial t}+u_\alpha\frac{\partial}{\partial x_\alpha}, \quad H^2 = H_\alpha H_\alpha,$

and repeated indices in the same term indicate summation of the terms as the dummy index runs through 1, 2 and 3. The vector forms of equations (1) and (2) will be useful. These are

$$\rho \frac{D}{Dt}\mathbf{v} = -\operatorname{grad} p + \rho \mathbf{X} + \mu \mathbf{j} \times \mathbf{H}, \quad 4\pi \mathbf{j} = \operatorname{curl} \mathbf{H}, \tag{1a}$$

and

$$\frac{\partial \mathbf{H}}{\partial t} = \operatorname{curl}(\mathbf{v} \times \mathbf{H}), \tag{2a}$$

in which $\mathbf{v}$ is the velocity, $\mathbf{X}$ the body force, $\mathbf{H}$ the field, and $\mathbf{j}$ the current density.

The equation of continuity is

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_\alpha}{\partial x_\alpha} = 0, \tag{3}$$

in which ρ need not be constant. Since the fluid is incompressible,

$$D\rho/Dt = 0, \tag{4}$$

and (3) can be written as

$$\partial u_\alpha/\partial x_\alpha = 0. \tag{5}$$

There is also an equation of continuity of the magnetic field, which is

$$\partial H_\alpha/\partial x_\alpha = 0. \tag{6}$$

This is in effect an initial condition which persists by virtue of equation (2a).

2. Steady magnetically reinforced flows

A steady magnetically reinforced flow is a steady flow in which the streamlines and magnetic lines are coincident. It will now be shown that, provided the fluid is inviscid, incompressible, and infinitely conducting, the totality of solutions for such flows is exactly the same as the totality of solutions for steady flows in the absence of a magnetic field. In other words, a steady magnetically reinforced flow is equivalent to and has the same pattern as some steady flow of an ideal fluid (which may be non-homogeneous) in the absence of a magnetic field. Note that for an infinitely conducting fluid the magnetic lines move with the fluid. Hence if streamlines and magnetic lines are coincident far upstream in a steady flow, they coincide everywhere, except possibly in regions of closed streamlines.

The above-stated theorem will now be proved. Since the flow is steady,

$$D/Dt = u_\alpha \, \partial/\partial x_\alpha,$$

and equations (1), (2) and (4) become

$$\rho u_\alpha \frac{\partial u_i}{\partial x_\alpha} = -\frac{\partial}{\partial x_i}\left(p + \frac{\mu}{8\pi} H^2\right) + \rho X_i + \frac{\mu}{4\pi} H_\alpha \frac{\partial H_i}{\partial x_\alpha}, \tag{1b}$$

$$u_\alpha \, \partial H_i/\partial x_\alpha = H_\alpha \, \partial u_i/\partial x_\alpha, \tag{2b}$$

and

$$u_\alpha \, \partial \rho/\partial x_\alpha = 0. \tag{4b}$$

The equations for streamlines are

$$\frac{dx_1}{u_1} = \frac{dx_2}{u_2} = \frac{dx_3}{u_3}, \tag{7}$$

the solutions of which are

$$\psi(x_1, x_2, x_3) = C \quad \text{and} \quad \chi(x_1, x_2, x_3) = C'. \tag{8}$$

The intersections of the two families of surfaces represented by (8) as each of C and C' takes on different values are the streamlines, which are therefore imbedded in the ψ-surfaces (on which ψ = constant) and χ-surfaces. Since

$$\partial\psi/\partial x_i \quad (i = 1, 2 \text{ and } 3)$$

are the direction numbers for the normal to a ψ-surface,

$$u_\alpha(\partial\psi/\partial x_\alpha) = 0. \tag{9}$$

Similarly,

$$u_\alpha(\partial\chi/\partial x_\alpha) = 0. \tag{10}$$

The velocity is then given by

$$\mathbf{v} = \text{grad}\,\psi \,\mathbf{x}\, \text{grad}\,\chi, \tag{11}$$

for more detailed developments, see Yih (1957). But the development of (11) is very old, and the present writer certainly does not claim priority in its discovery.

Since the streamlines and magnetic lines are coincident, the most general† relationship between u_i and H_i is‡

$$H_i = \lambda(\psi, \chi)\, u_i, \tag{12}$$

in which λ is an arbitrary function of ψ and χ. Note that (2*b*) is then automatically satisfied, because both sides are equal to $\lambda u_\alpha(\partial u_i/\partial x_\alpha)$. The other terms on the right-hand side are $u_i u_\alpha(\partial\lambda/\partial x_\alpha)$ and these are zero as a result of (9) and (10). Equation (1*b*) becomes, again by virtue of (9) and (10),

$$\left(\rho - \frac{\mu}{4\pi}\lambda^2\right) u_\alpha \frac{\partial u_i}{\partial x_\alpha} = -\frac{\partial}{\partial x_i}\left(p + \frac{\mu}{8\pi}H^2\right) + \rho X_i. \tag{1c}$$

Note in passing that from either (1*b*) or (1*c*) the Bernoulli equation *along a streamline*

$$\tfrac{1}{2}\rho u_\alpha u_\alpha - (\mu/8\pi)\,H^2 = -p - (\mu/8\pi)\,H^2 - \rho\Omega + C(\psi, \chi), \tag{13}$$

or

$$\tfrac{1}{2}\rho u_\alpha u_\alpha + p + \rho\Omega = C(\psi, \chi), \tag{14}$$

can be derived, provided the body force has a potential Ω.

Up to this point the development is the same as that given by Grad§ (1960), who took (with a slight difference in notation)

$$\rho^* = \rho - \frac{\mu}{4\pi}\lambda^2 \quad \text{and} \quad p^* = p + \frac{\mu}{8\pi}H^2,$$

observed that

$$u_\alpha(\partial\rho^*/\partial x_\alpha) = 0, \tag{4c}$$

and concluded that the resulting equations

$$\rho^* u_\alpha(\partial u_i/\partial x_\alpha) = -\partial p^*/\partial x_i \tag{1d}$$

† The dependence of λ only on ψ and χ is dictated by (5) and (6).

‡ This relationship was already recognized by Grad (1960), whose work will be discussed in the following paragraph. Goldsworthy (1961) also recognized it. But he took λ to be constant in further discussion.

§ I am indebted to my colleague W. W. Willmarth for calling my attention to Grad's paper after this work was done.

govern the flow of a non-homogeneous fluid in the absence of a magnetic field. However, the body force has been neglected in Grad's equation, or (1d). If the term ρX_i is included in (1d), his conclusion does not go through. Furthermore, if body force is indeed neglected, his conclusion does not go far enough. For we can define (Yih 1958)

$$u_i' = (\rho^*/\rho_0)^{\frac{1}{2}} u_i,$$

and by virtue of (4c) write (1d) and (5) as

$$\rho_0 u_\alpha'(\partial u_i'/\partial x_\alpha) = -\partial p^*/\partial x_i,$$

and

$$\partial u_\alpha'/\partial x_\alpha = 0,$$

thus reducing the equations to those governing the flow of an incompressible fluid of *homogeneous* density. The boundary conditions present no difficulty, as will be shown for the general case, in which body force is taken into account.

Now let

$$Q = 1 - \frac{\mu}{4\pi\rho}\lambda^2, \quad u_i' = |Q|^{\frac{1}{2}} u_i, \quad p' = \frac{Q}{|Q|}\left(p + \frac{\mu}{8\pi}H^2\right), \quad \Omega' = \frac{Q}{|Q|}\Omega. \tag{15}$$

If $Q = 0$, we take $Q/|Q|$ to be 1, although the value -1 will do as well. By virtue of (9) and (10), equations (1c) can be written as

$$\rho u_\alpha' \frac{\partial u_1'}{\partial x_\alpha} = -\frac{\partial p'}{\partial x_i} - \rho\Omega'. \tag{16}$$

Furthermore,

$$\partial u_\alpha'/\partial x_\alpha = 0 \tag{17}$$

holds by virtue of (5), (9) and (10), and

$$u_\alpha'(\partial\rho/\partial x_\alpha) = 0 \tag{18}$$

obviously holds by virtue of (4b). But (16), (17) and (18) are the equations governing steady flows of an incompressible and inviscid fluid in the absence of a magnetic field.

The boundary conditions need to be considered. If solid boundaries are present, the kinematic conditions at these boundaries are identical for the original flow (u_i-field) as for the associated flow (u_i'-field). Dynamical boundary conditions in the original flow can always be translated into similar ones in the associated flow.

The change of sign of Q in the field does not present any difficulty. Consider a surface S separating a region of positive Q from a region of negative Q. On the surface, $Q = 0$, and

$$\{p + (\mu/8\pi)H^2\} + \rho\Omega = C, \tag{19}$$

corresponding to

$$p' + \rho\Omega' = C. \tag{20}$$

The condition $Q = 0$ also corresponds to

$$u_\alpha' u_\alpha' = 0 \tag{21}$$

on S. Since the flow u_i' is not irrotational, (21) is not impossible. The two parts of the associated flow on opposite sides of S are all governed by (16), (17) and (18), and on S, (20) and (21) hold. The sign of C (though not its magnitude) changes abruptly. But this does not violate the equality of

$$p + (\mu/8\pi)H^2$$

at S and on both sides of S.

If λ and ρ are constant and Q is zero all over the field, so that

$$u_i = (\mu/4\pi\rho)^{\frac{1}{2}} H_i, \tag{22}$$

the equations of motion are automatically satisfied and the pressure p can be evaluated from (19).

3. Establishment of irrotational and current-free flows

Equation (1*a*) indicates that if the magnetic field is current-free, irrotational flow is possible. If, further, the streamlines and magnetic lines are coincident, (2*a*) indicates that the magnetic field is steady. The velocity field may be steady or not; both steady and unsteady irrotational flows are possible, so long as $\mathbf{v}$ is parallel to $\mathbf{H}$, and $\mathbf{j}$ is zero.

For convenience, flows for which streamlines and magnetic lines are coincident and both the velocity field and the magnetic field are irrotational will be called I^2-flows. Steady I^2-flows past cylindrical bodies have been studied by Sears & Resler (1959), who discussed not only such flows but also flows in which $\mathbf{v}$ is not parallel to $\mathbf{H}$ (see also Sears 1961). We are concerned here only with I^2-flows. Sears & Resler did not state how their steady I^2-flow was established, and Stewartson (1960), assuming† that establishment from an initial uniform field was implied in the work of Sears & Resler, showed that if U_0 is the velocity of the fluid and H_∞ the magnetic field strength at infinity and if

$$U_0(4\pi\rho/\mu)^{\frac{1}{2}} \leqslant H_\infty, \tag{23}$$

the final steady flow is a slug flow predicted with a simple development by Yih (1959) for an infinitely conducting fluid. We shall show here by a very simple argument that if H is intially uniform and has the single component $H_1 = H_\infty$, and if the fluid has only one component $u_1 = U(t)$ at infinity, which increases from zero to U_0, steady I^2-flows cannot be the final state. On the other hand, if a current-free magnetic field identical with the one in a steady I^2-flow is set up *before* $U(t)$ increases (or before the body moves), the final steady flow must be an I^2-flow. If the body is a sphere, such a field can be set up by superposing on the uniform magnetic field H_∞ a field due to a magnetic dipole at the centre of the sphere, with proper strength and the axis pointing to a direction opposite to that of H_∞. The dipole can of course be replaced by the sort of surface current Sears & Resler described. In the following we shall consider the motion *relative* to the body. It can be shown that the kinematical problem is quite the same, whether the absolute or the relative motion is considered, although this may not be immediately evident as in the non-magnetic case (see the discussion in the Appendix).

The discussion will not be limited to two-dimensional flows. First, note that flows governed by (1), (2), (4), (5) and (6) are all reversible. Suppose that in $-T \leqslant t \leqslant 0$ the flow is established as $U(t)$ increases from zero to U_0, and the velocity and magnetic fields are given by U_i and H_i. If $U(t)$ now decreases from U_0 to zero according to the reversed schedule

$$U(t) = -U(-t) \quad (t > 0), \tag{24}$$

† However, he was probably aware that an I^2-flow can be established from an initially streaming magnetic field. See also a further paper by him (1963).

then for $t > 0$, the governing equations are exactly satisfied if

$$u_i(t) = -u_i(-t), \quad H_i(t) = H_i(-t). \tag{25}$$

For the effect of the discontinuity in velocity implied in (25), see the Appendix.

Now consider the establishment of flow in the period $-T \leqslant t \leqslant 0$, with a streaming current-free magnetic field already at $t = -T$, and with $U(t)$ varying from zero at $t = -T$ to U_0 at $t = 0$. The flow is initially current-free. Hence the term $\mu \mathbf{j} \times \mathbf{H}$ in ($1a$) is zero, and the initial motion will be irrotational, with the streamlines coinciding with the magnetic lines. Then ($2a$) states that the magnetic field will be stationary, and therefore will continue to be current-free. Further increase of $U(t)$ will then not affect the flow pattern, and will give a fluid velocity at every point proportional to the velocity of the final steady flow, the coefficient of proportionality being $U(t)/U_0$. Thus the final I^2-flow will be reached through intermediate I^2-flows. Now if, starting from the steady flow so obtained, we reverse the flow according to (24), not only do we know that (25) will satisfy the governing equations, but exactly the same argument as just put forth will show that the original (at $t = -T$) quiescent state with a streaming current-free magnetic field, *and no other state*, will be reached at $t = T$. If it had been possible to establish the steady state of I^2-flow at $t = 0$ from an initial quiescent state with a uniform magnetic field $H_1 = H_\infty$, $H_2 = 0 = H_3$, it should have been possible to recover it according to (25) as $U(t)$ varies from U_0 to zero according to (24) in the interval $T \geqslant t \geqslant 0$, contrary to the result established above. Hence it is impossible to establish a steady I^2-flow from a quiescent state with a uniform magnetic field. Although it would be more realistic to consider real fluids of finite viscosity and conductivity, discussion of the question of establishment within the framework of inviscid and infinitely conducting fluids by the use of the simple idea of reversibility perhaps throws some light on the matter.

4. A current-induced jet

Long (1960) has studied the steady axisymmetric motion of an infinitely conducting fluid. Cylindrical co-ordinates r, θ, and z will be used. The velocity components in the directions of increasing r, θ, and z will be denoted by u, v, and w. The components of the magnetic field will be denoted by H_r, H_θ, and H_z. Following Long, we shall use

$$(f, g, h) = (\mu/4\pi\rho)^{\frac{1}{2}} (H_r, H_\theta, H_z). \tag{26}$$

From the equations of continuity for the flow and magnetic fields,

$$ur = -\psi_z, \quad wr = \psi_r, \tag{27}$$

$$fr = -\Lambda_z, \quad hr = \Lambda_r, \tag{28}$$

in which ψ is the Stokes stream function, Λ the corresponding function for the magnetic field, and subscripts denote partial differentiation. From ($2a$), with the left-hand side equal to zero, Long was able to show that

$$\Lambda = \Lambda(\psi) \tag{29}$$

and

$$(g - \Lambda' v)/r = K(\psi), \tag{30}$$

with Λ' indicating $d\Lambda/d\psi$. Equation (29) shows that projections of the streamlines and magnetic lines coincide in the (r, z) plane. Furthermore, from the θ-equation in (1*a*), again for steady flow, he was able to show

$$vr - \Lambda' gr = L(\psi). \tag{31}$$

Finally, from the other two equations of motion Long showed that

$$\psi_{zz} + \psi_{rr} - \frac{1}{r}\psi_r - \frac{\Lambda\Lambda}{1-\Lambda'^2}[(\psi_z)^2 + (\psi_r)^2] + \frac{A'}{2(1-\Lambda'^2)} + \frac{B'r^4}{2(1-\Lambda'^2)} = \frac{M(\psi)\,r^2}{1-\Lambda'^2}, \tag{32}$$

in which $$A = L^2/(1-\Lambda'^2), \quad B = K^2/(1-\Lambda'^2). \tag{33}$$

Now (29) indicates that, although the magnetic lines and streamlines do not necessarily coincide, their projections in the (r, z) plane do. Thus the flow is magnetically reinforced as far as the r and z components of the velocity and of the magnetic field are concerned. From the results obtained in § 2, we expect that (32) can be considerably simplified by a transformation similar to the second equation in (15). Realizing that Λ' is really just the $(\mu/4\pi\rho)^{\frac{1}{2}}\lambda$ in (15), we make the transformation

$$d\Psi = (1-\Lambda'^2)^{\frac{1}{2}}\,d\psi, \tag{34}$$

which reduces (32), after it has been multiplied by $(1-\Lambda'^2)^{\frac{1}{2}}$, to

$$\Psi_{zz} + \Psi_{rr} - \frac{1}{r}\Psi_r + \frac{1}{2}\frac{dA}{d\Psi} + \frac{1}{2}\frac{dB}{d\Psi}r^4 = N(\Psi)\,r^2, \tag{35}$$

in which $$N = M(1-\Lambda'^2)^{-\frac{1}{2}}.$$

That (35) is considerably simpler than (32) is quite evident.

Long gave two explicit linear cases of (32). The first is for the conditions $v_0 = 0$, $w_0 =$ const. $g_0 = 0$, $h_0 =$ const. at $z =$ infinity. In this case (32) reduces to

$$\psi_{zz} + \psi_{rr} - (1/r)\,\psi_r = 0.$$

The second is for the same conditions at infinity except $v_0 = 0$, which is replaced by $v_0 = \omega r$ (solid rotation). In this case

$$\psi_{zz} + \psi_{rr} - \frac{1}{r}\psi_r + \sigma^2\psi = \tfrac{1}{2}\sigma^2 w_0 r^2, \quad \sigma = \frac{2\omega}{w_0[1-(h_0/w_0)^2]}. \tag{36}$$

Long stated that other linear cases could be found by a procedure he used to deal with the vorticity equation for stratified flow in a gravitational field (Long 1958). But that procedure was quite labourious even for the equation to which it was applied, and will certainly be much more so if one attempts to apply it to (32). Furthermore, it cannot be used to find the linear cases exhaustively. We shall use the same approach as used by Yih (1960) for discovering the linear cases of steady stratified flows in a gravitational field. Starting with (35), the possible linear cases are simply the cases in which

$$\frac{1}{2}\frac{dA}{d\Psi} = a\Psi + b, \quad \frac{1}{2}\frac{dB}{d\Psi} = c\Psi + d, \quad N = m\Psi + n, \tag{37}$$

in which the lower-case letters are constants. Given *any* $\Lambda(\psi)$, the first two equations in (37) specify $L(\psi)$ and $K(\psi)$, and therefore v_0 and g_0. With (37) substituted in (35), the upstream condition is then determined by solving Ψ as a function of r far upstream. The function ψ is then also known as a function of r.

For instance, if $a = k^2, b = 0, c = 0, d = 0$, and $m = 0$ for any Λ', (35) becomes

$$\Psi_{zz}+\Psi_{rr}-(1/r)\Psi_r+k^2\Psi = nr^2, \tag{38}$$

which gives

$$\Psi_\infty = \beta r^2, \quad \beta = n/k^2. \tag{39}$$

If, further,

$$g = \Lambda' v, \tag{40}$$

then $B = 0$, and far upstream

$$L = (1-\Lambda'^2)\,vr, \quad (1-\Lambda'^2)\,v^2r^2 = a\Psi^2 = (k\beta r^2)^2, \tag{41}$$

the latter of which specifies v_0 (at z = infinity) if Λ' is specified, upon use of (39). The solution of (38) is of the form

$$\Psi = \beta r^2+\sum_{n=1}^{\infty} A_n rJ_1(k_n r)\exp\{\pm\sqrt{(k_n^2-k^2)}\,z\}, \tag{42}$$

in which $J_1(k_n R) = 0$ if the flow is within a cylinder of radius R.

We shall present an interesting case of an axisymmetric flow in a cylinder of radius R into a point sink (of flow as well as electricity) at $z = 0$. Since the flow is symmetric with respect to the plane $z = 0$ it is sufficient to consider the *lower* half of the cylinder. The case described by

$$w = W, \quad h = h_0, \quad v = \omega r, \quad g = \gamma r = 2\pi j_0(\mu/4\pi\rho)^{\frac{1}{2}}\, r \tag{43}$$

at $z = -\infty$ corresponds to a uniform flow, a uniform magnetic field, a solid rotation, and a uniform current density j_0 at $z = -\infty$. For definiteness, let

$$\Lambda' = h_0/W = \alpha < 1.$$

The conditions at infinity give $\psi = \frac{1}{2}Wr^2$ and B = const. so that $B' = 0$. As to A,

$$vr-\Lambda' gr = (\omega-\alpha\gamma)\,r^2 = \frac{2(\omega-\alpha\gamma)}{W}\psi, \tag{44}$$

so that

$$A = 4\left(\frac{\omega-\alpha\gamma}{W}\right)^2(1-\alpha^2)^{-1}\psi^2. \tag{45}$$

Since Λ' = const. we can use (32) directly, giving

$$\psi_{zz}+\psi_{rr}-\frac{1}{r}\psi_r+k^2\psi = \frac{k^2}{2}Wr^2, \tag{46}$$

in which the right-hand side has been determined from the upstream condition, and

$$k = 2(\omega-\alpha\gamma)/W(1-\alpha^2). \tag{47}$$

The solution of (46) is

$$\psi = \frac{W}{2}r^2+\sum_{n=1}^{\infty} A_n rJ_1(k_n r)\exp(k_n^2-k^2)^{\frac{1}{2}}z, \tag{48}$$

in which

$$J_1(k_n R) = 0, \quad A_n = \frac{W}{k_n J_0^2(k_n R)}. \tag{49}$$

The coefficients A_n are determined from the condition that $\psi = \frac{1}{2}WR^2$ at $z = 0$. The first eigenvalue $k_1 R$ is 3·831. If

$$k^2R^2 < (3{\cdot}831)^2, \tag{50}$$

the upstream condition is satisfied by (48). The solution is similar to the one obtained by Long (1956) for a rotating fluid. It is characterized by a jet more and more concentrated along the centreline as k^2R^2 increases toward $(3\cdot831)^2$, and by

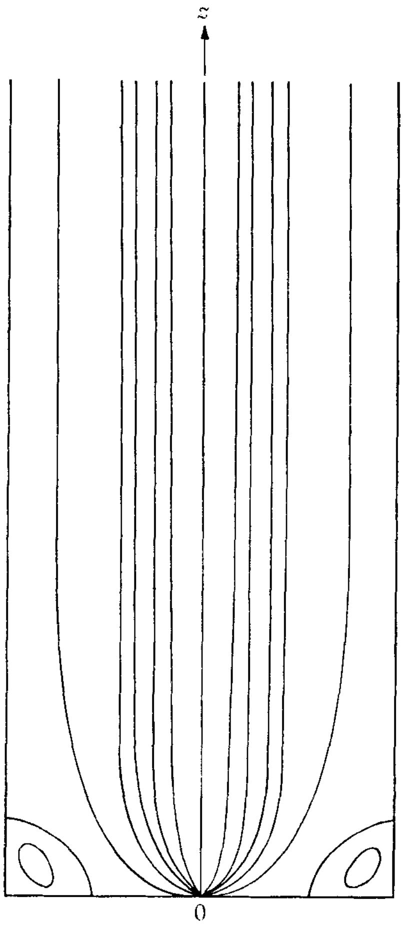

FIGURE 1. A sketch of the flow pattern for a current-induced jet. The plane $z = 0$ can either be a solid boundary or a plane of symmetry separating the flow shown from its mirror image described by (48).

a ring eddy on each side of the plane $z = 0$. The mirror image of the flow described by (48) is shown in figure 1. If

$$k^2R^2 > (3\cdot831)^2, \tag{51}$$

the upstream condition is no longer satisfied by (48), and vortex sheets (and possibly current sheets) can be expected to occur, in analogy with stratified flow (see Debler 1959).

Now the form of k^2 given in (47) allows one to say that k^2 will be large if

(1) $\alpha\gamma R$ is large relative to W (strong current),

(2) ωR is large relative to W (strong rotation),

(3) α is near unity.

It is interesting that between the limiting cases (1) and (2), the effects of rotation and current can be compensative, and the signs of $\alpha\gamma$ (or αj_0) and ω are all important in the intermediate range. Item (3) is pertinent to the discussion given by Long (1960) at the end of his paper. We have assumed the upstream condition to be undisturbed. This corresponds to $\alpha^2 < 1$ according to Long's conclusion. No explanation has been given as to why the blocking effect, which occurs at large k^2, should be critical as α^2 approaches unity. An explanation for the case discussed by Long and described by (36) here is as follows: As α^2 approaches 1, W approaches h_0, which is the phase velocity c *and* the group velocity c_g of waves at zero wavelength. (For $\alpha < 1$, $c_g > c$; see Long 1960.) Therefore as $\alpha^2 \to 1$ from below, *even* the short waves are able to travel upstream. The shadow effect or blocking effect is then dominant. This explanation for the phenomenon governed by (36) is applicable also to the phenomenon under study here, governed by (46). Mathematically, the homogeneous part of (46) for large k^2 is similar to the equation governing short acoustic waves. The shadow effect of high pitch in sound is well known, and is mathematically analogous to the blocking effect of high k^2 in the present problem.

Note that the jet phenomenon can be caused by j_0 alone, with $\omega = 0$, whereas without an electrical current no such phenomenon can occur without rotation, as indicated by (36).

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Appendix

The equivalence of the absolute motion and the motion relative to the body is based on (2) or (2a). If u_i is the ith component of the velocity relative to the body, then the absolute velocity has the components

$$-U(t)+u_1, \quad u_2 \quad \text{and} \quad u_3,$$

in which $U(t)$ is the speed of the body, which moves in the direction of decreasing x_1. Now if u_i is proportional to H_i, $(2a)$ becomes, after utilization of (6),

$$\frac{\partial \mathbf{H}}{\partial t} - U\frac{\partial \mathbf{H}}{\partial x} = 0,$$

which states that the magnetic field moves with the body. If it is initially irrotational, it will continue to be. The irrotationality of the flow then follows from the fact that $\mathbf{j} = 0$, and the Helmholtz–Kelvin conservation theorems.

However, displacement current has so far not been considered, and my colleague, Dr J. D. Murray, has pointed out to me that the sudden reversal of velocity proposed in § 3 may result in infinite displacement currents. If displacement currents are to be included, $(1a)$ should be replaced by

$$4\pi\mathbf{j} = \operatorname{curl}\mathbf{H} - \frac{\partial(\epsilon\mathbf{E})}{\partial t}, \tag{A 1}$$

in which ϵ is the dielectric constant. The Ohm's law is still

$$\mathbf{j} = \sigma(\mathbf{E} + \mu\mathbf{v}\times\mathbf{H}), \tag{A 2}$$

in which σ is the electrical conductivity. The Maxwell equation

$$\mu\frac{\partial \mathbf{H}}{\partial t} = -\operatorname{curl}\mathbf{E} \tag{A 3}$$

is still valid. Eliminating $\mathbf{E}$ and $\mathbf{j}$ from the three preceding equations, we have

$$\frac{\partial \mathbf{H}}{\partial t} = \operatorname{curl}(\mathbf{v}\times\mathbf{H}) - \eta\operatorname{curl}\operatorname{curl}\mathbf{H} - \eta\epsilon\mu\frac{\partial^2\mathbf{H}}{\partial t^2}, \tag{A 4}$$

if ϵ is assumed constant. Now for an infinitely conducting fluid, $\eta = 0$, and the last two terms vanish provided $\nabla^2\mathbf{H}$ and $\partial^2\mathbf{H}/\partial t^2$ are not infinite. Then $(2a)$ is recovered, and the argument in § 3 can again be applied to reach the conclusion of the persistence of the I^2-state.

To ensure that $\partial^2\mathbf{H}/\partial t^2$ be not infinite, it is necessary to have a smooth reversal instead of the sudden reversal proposed in § 3, if displacement currents are taken into account. To do so, assume that an I^2-flow is established in any way whatsoever in the interval $-T < t < -t_1$, in which $U(t)$ increases from zero to U_0, and let $U(t)$ decrease smoothly but otherwise in any arbitrary manner from U_0 to zero in the interval $-t_1 < t < 0$. At $t = 0$ we have a quiescent fluid with a streaming irrotational magnetic field, according to the arguments in § 3. We then apply (25). On the one hand, the reversibility of the flow and the magnetic field demand that the initial ($t = -T$) condition be retrieved at $t = T$. On the other hand, the I^2-state must persist from the moment $t = -t_1$, as argued in the preceding paragraph and in § 3. Hence a contradiction would be reached if the I^2-flow had been established from an initially uniform field at $t = -T$. Note that in this argument the smooth slowing down in $-t_1 < t < 0$ is introduced merely to remove the discontinuity which would be implied in (35) if t_1 were taken to be zero, as in § 3. The argument is quite unimpaired by extending the history of the motion to $t = 0$ before reversing it.